# Comments on quasiparticle models of quark-gluon plasma

Vishnu M. Bannur
*Department of Physics,*
*University of Calicut, Kerala-673 635, India.*

November 14, 2006

**Abstract**

Here we comment on the thermodynamic inconsistency problem and the reformulation of statistical mechanics of widely studied quasiparticle models of quark-gluon plasma. Their starting relation, the expression for pressure itself is a wrong choice and lead to thermodynamic inconsistency and the requirements of the reformulation of statistical mechanics. Hence we revise the model using the standard statistical mechanics and is thermodynamically consistent. We also show that the other quasiparticle models may be obtained from our general formalism as a special case under certain restrictive condition. Further, as an example, we applied our model to explain the nonideal behaviour of gluon plasma and obtained a remarkable good fit to the lattice results by adjusting just a single parameter.

**PACS Nos :** 12.38.Mh, 12.38.Gc, 05.70.Ce, 52.25.Kn

**Keywords :** Equation of state, quark-gluon plasma, quasiparticle quark-gluon plasma.

# 1 Introduction :

The quasiparticle quark-gluon plasma (qQGP) is a phenomenological model, with few fitting parameters, widely used to describe the nonideal behaviour of quark-gluon plasma (QGP). It was first proposed by Peshier *et. al.* [1] to explain the equation of state (EoS) of QGP from lattice gauge theory (LGT) simulation of quantum chromodynamics (QCD) at finite temperature [2]. The model, however, failed [4] to explain the more accurate, recent lattice data [3]. Further more, Gorenstein and Yang [5] pointed out that the model is thermodynamically (TD) inconsistent. This TD inconsistency problem was remedied by them by introducing a temperature dependent vacuum energy and forced it to cancel the TD inconsistent term, which they called as the reformulated statistical mechanics. It is still not clear what is the physics or origin of this constraint which they all [5, 6, 7, 8] called TD consistency check. Here we show that the whole exercise is unnecessary and following the standard statistical mechanics (SM), we propose a new qQGP model which contains a single phenomenological parameter. Our model is TD consistent and explains lattice data very well.

# 2 Our model of qQGP:

Let us start with the work of Peshier *et. al.* [1] on gluon plasma (GP). They derived all TD quantities from the pressure, $P$, which they assumed as

$$\frac{P\,V}{T} = -\sum_{k=0}^{\infty} \ln(1 - e^{-\beta\epsilon_k}) \ , \tag{1}$$

where the right hand side is the logarithm of the grand partition function, $Q_G(T)$, and $\epsilon_k$ is the single particle energy of quasi-gluon, i.e, gluon with temperature dependent mass, given by,

$$\epsilon_k = \sqrt{k^2 + m^2(T)} \ ,$$

where $k$ is momentum and $m$ is mass. $\beta$ is defined as $1/T$. The expression for pressure is similar to that of ideal gas with temperature dependent mass and hence let us denote it as $P_{id}$. We want to stress that this assumption itself is the root cause of TD inconsistency and hence the reformulation of SM by Gorenstein and Yang [5]. Generally, in grand canonical ensemble (GCE), energy ($E_r$) and number of particles ($N_s$) fluctuate, but temperature ($T$) and the chemical potential ($\mu$) are fixed. Hence, the average energy ($U$) and average number of particles ($N$) are defined and may be related to the grand partition function or q-potential,

$$q \equiv \ln Q_G = \ln(\sum_{s,r} e^{-\beta E_r - \alpha N_s}) = \mp \sum_{k=0}^{\infty} \ln(1 \mp z\, e^{-\beta\epsilon_k}) \ , \tag{2}$$

where $\mp$ for bosons and fermions and $z \equiv e^{\mu/T} = e^{-\alpha}$ is called fugacity. The average energy $U$ is defined as,

$$U \equiv < E_r > = \frac{\sum_{s,r} \; E_r \, e^{-\beta E_r - \alpha N_s}}{Q_G} = -\frac{\partial}{\partial \beta} \ln Q_G = \sum_k \frac{z \, \epsilon_k e^{-\beta \epsilon_k}}{1 \mp z \, e^{-\beta \epsilon_k}} \; . \tag{3}$$

Note that the partial differentiation with repect to $\beta$ above is just a mathematical trick to express $U$ in terms of sum over single particle energy levels, $\epsilon_k$, making use of Eq. (2). While differentiating, indirect dependence of $\beta = 1/T$ in the fugacity, $z$, and mass, $m(T)$, must be ignored by definition. Otherwise, we will not get back $< E_r >$. Similarly, the average density $N$ is defined as,

$$N \equiv < N_s > = \frac{\sum_{s,r} \; N_s \, e^{-\beta E_r - \alpha N_s}}{Q_G} = -\frac{\partial}{\partial \alpha} \ln Q_G = z \frac{\partial}{\partial z} \ln Q_G = \sum_k \frac{z \, e^{-\beta \epsilon_k}}{1 \mp z \, e^{-\beta \epsilon_k}} \; , \tag{4}$$

These (Eqs. (3), (4)) are the standard relations [9] of $U$ and $N$ to the partition function, which is valid even for quasiparticle with ($T$,$\mu$) dependent mass by the definition of averages. Here for GP we have $\mu = 0$ or $z = 1$. Next, pressure may be obtained by two methods. In method-I, one starts from $U$ and using TD relation,

$$\varepsilon \equiv \frac{U}{V} = T \frac{\partial P}{\partial T} - P \; , \tag{5}$$

and on integration, one gets pressure which is the procedure we follow here. In method II, again following the standard text books on SM [9], we can relate $P$ to q-potential as follows. The variation in q-potential due to variations in it's dependence, namely $T$, $\mu$ and volume $V$, specifying the macro-state of GCE system, is,

$$\delta q = \frac{1}{Q_G} \left[ \sum_{r,s} e^{-\beta (E_r - \mu N_s)} \; (-E_r \, \delta\beta - \beta \, \delta E_r + N_s \, \delta(\beta\mu) \,) \right] \; . \tag{6}$$

Now, compared to the text books results, we have extra term coming from $\delta E_r$ due to temperature dependent mass and then using the definition of averages, so on, we get,

$$\frac{P \, V}{T} = \mp \sum_{k=0}^{\infty} \ln(1 \mp z \, e^{-\beta \epsilon_k}) + \int d\beta \, \beta \, \frac{\partial m}{\partial \beta} \; < \frac{\partial E_r}{\partial m} > \; . \tag{7}$$

Therefore we see that $P$ is not just equal to $P_{id}$, but there is an extra term. This extra term ensure TD consistency of the relation as follows. From above $P$, on differentiating with respect to $T$ for a system with $\mu = 0$ or $z = 1$,

$$\frac{\partial P}{\partial T} = \frac{P}{T} + \frac{\varepsilon}{T} - \frac{1}{V} \; < \frac{\partial \epsilon_k}{\partial T} > + \frac{1}{V} \; < \frac{\partial E_r}{\partial T} > \tag{8}$$

where the last two terms exactly cancels (following the trick used in deriving Eq, (3)) and hence the thermodynamic relation, Eq. (5), is obeyed as expected.

Further, this $P$ is also consistent with the $P$ obtained from $U$ through TD relations which may be shown as follows. The Eq. (7) may be further simplified by evaluating $< \frac{\partial E_r}{\partial m} >$ and taking continuum limit, for a system with $\mu = 0$, as,

$$\frac{P}{T} = \mp \frac{g_f}{2\pi^2} \int_0^\infty dk\, k^2 \, \ln(1 \mp e^{-\beta \epsilon_k}) + \int d\beta\, \beta \frac{g_f}{2\pi^2}\, m\, \frac{dm}{d\beta} \int_0^\infty dk \frac{k^2}{\epsilon_k \, (e^{\beta \epsilon_k} \mp 1)} \;, \tag{9}$$

which on further algebra, reduces to,

$$\frac{P}{T} = \frac{g_f}{2\,\pi^2} \left[ T^3 \sum_{l=1}^{\infty} (\pm 1)^{l-1} \frac{1}{l^4} \, (\beta\, m\, l)^2 \, K_2(\beta\, m\, l) \right.$$

$$\left. + \int d\beta \frac{\beta}{m} \frac{\partial m}{\partial \beta} \frac{1}{\beta^4} \sum_{l=1}^{\infty} (\pm 1)^{l-1} \frac{1}{l^4} \, (\beta\, m\, l)^3 \, K_1(\beta\, m\, l) \right] \;, \tag{10}$$

where $g_f$ is the internal degrees of freedom and $K_1$, $K_2$ are modified Bessel functions. Using the recurrence relations of Bessel functions and on integration by parts, above equation may be further simplified to get,

$$\frac{P}{T} = \frac{P_0}{T_0} - \int_{\beta_0}^{\beta} d\beta \, \varepsilon \;, \tag{11}$$

where $\varepsilon$ is the energy density and $P_0$ is the pressure at some temperature $T_0$ or $\beta_0$. This equation is nothing but the thermodynamic relation, Eq. (5). Therefore, Gorenstein and Yang's starting argument that above two methods give different $m(T)$, etc. does not exist now by using our derived expression for $P$, instead of the assumption [1, 5].

## 3 Question of vacuum energy $B(T)$ :

After the reformulation of SM by Gorenstein and Yang, almost all study in qQGP is based on their TD consistency relation, related to vacuum energy $B(T)$. Different authors call and interpret $B(T)$ in a different way, like vacuum energy, background field, bag pressure, etc. Out of them reasonable interpretation of $B(T)$ for a deconfined state, QGP, may be vacuum or zero point energy. But, by definition of quasiparticle, whole thermal energy is used to excite these quasiparticles. So quasiparticles are excitations above the ground state or vacuum state which may not depend on temperature or chemical potential. This is our assumption. As noted earlier, we also don't have TD inconsistency in our model.

In fact, when we redo our derivation of pressure with vacuum or zero point energy in single particle energy, like in Ref. [5], Eq. (9) is modified as,

$$P = P_{id} - B(T) + T \left( \int_{T_0}^{T} \frac{d\tau}{\tau} \left[ \frac{g_f}{2\pi^2}\, m\, \frac{dm}{d\tau} \int_0^\infty dk \frac{k^2}{\epsilon_k \, (e^{\epsilon_k/\tau} \mp 1)} + \frac{\partial B}{\partial \tau} \right] \right) \;, \tag{12}$$

and the energy density,

$$\varepsilon = \varepsilon_{id} + B(T) \ . \tag{13}$$

where $\varepsilon_{id}$ is the expression for $\varepsilon$ similar to ideal gas. Again it is easy to show that above $P$ and $\varepsilon$ obey TD relation Eq. (5) and hence TD consistent. Interesting thing here is that the TD consistency relation [5], used in other qQGP models, is nothing but a restrictive condition that the terms inside the square bracket in Eq. (12) is zero. At present it is not clear what is the physical origin of this constraint. Note that without this constraint, so called TD consistency relation, our system is TD consistent even with the zero-point energy contribution, $B(T)$. But $B(T)$ is an unknown function and we speculate that it may depend on effects of hadronic states, resonances, etc. and may be relevent at the transition point. In our study of gluon plasma here, we neglect all these effects and consider only the thermal properties of gluons. Hence we take $B(T) = 0$ and we get a very good result except at very close to the transition temperature, i.e, for $T < 1.2T_c$.

It should also be pointed out that, in Ref. [6], authors claim that their qQGP model with $B(T)$, reproduces QCD perturbative results to order $\alpha_s$ which they claim that it comes from $B(T)$. Probably by mistake they have not included the terms of same order in $m$ or $\alpha_s$, coming from other term, the partition function, as shown below. From Ref. [6], Eq. (3) is,

$$P(T) = \frac{d}{6\pi^2}\int_{k=0}^{\infty} dk \, \frac{1}{(e^{\beta\sqrt{k^2+m^2(T)}} - 1)} \, \frac{k^4}{\sqrt{k^2 + m^2(T)}} - B(T) \ , \tag{14}$$

which on high temperature limit gives,

$$P = P_{SB}\left[1 - \frac{15}{4\pi^2}(\frac{m}{T})^2 - \frac{15}{8\pi^2}(\frac{m}{T})^2\right] \ . \tag{15}$$

The last term comes from $B(T)$ as mensioned in Ref. [6], but second term comes from the partition function which they missed and came to the wrong conclusion. So it invalidate their claim that their qQGP with $B(T)$ is justifiable from QCD point of view. Therefore, our model also may not reduce to QCD results of order $\alpha_s$ at high temperature limit, probably, because the quasiparticle models of QGP are phenomenological models based on the collective effects of plasma.

One more comment on, so called the additional requirement for TD consistency, Eqs. (21), (22) and (23) in Ref. ([5], is that they are not related. That is, it is not possible to get Eq. (23) from Eq. (22) by integration since there is another temperature dependence, coming from the distribution function, in addition to $m(T)$. The problem lies in the much discussed additional requirement relation, Eq. (21) or (14), in that paper. It need to be modified as

$$\left(\frac{\partial P}{\partial m}\right)_T = 0 \;\rightarrow\; \frac{dm}{dT}\left(\frac{\partial P}{\partial m}\right)_T = 0, \tag{16}$$

and with the assumption that $B(T)$ depends on $T$ only through $m(T)$, i.e. $B(m(T))$, in order to get Eq. (23) which fortunately respects TD consistency in their model. Therefore, their general additional requirement conditions, Eq. (14) in Ref. [5], is not enough for TD consistency in their model.

## 4 EoS of gluon plasma:

As an example, let us apply our model to GP. We first calculate the energy density, expressed in terms of $e(T) \equiv \varepsilon/\varepsilon_s$, and then obtain $P$ from TD relation, Eq. (5). So we have, from Eq. (3) after some algebra,

$$e(T) = \frac{15}{\pi^4}\sum_{l=1}^{\infty}\frac{1}{l^4}\left[(\frac{m_g\, l}{T})^3 K_1(\frac{m_g\, l}{T}) + 3\,(\frac{m_g\, l}{T})^2 K_2(\frac{m_g\, l}{T})\right] \tag{17}$$

where $\varepsilon_s$ is the Stefan-Boltzman gas limit of QGP, $m_g$ is the temperature dependent mass and $K_1$, $K_2$ are modified Bessel functions.

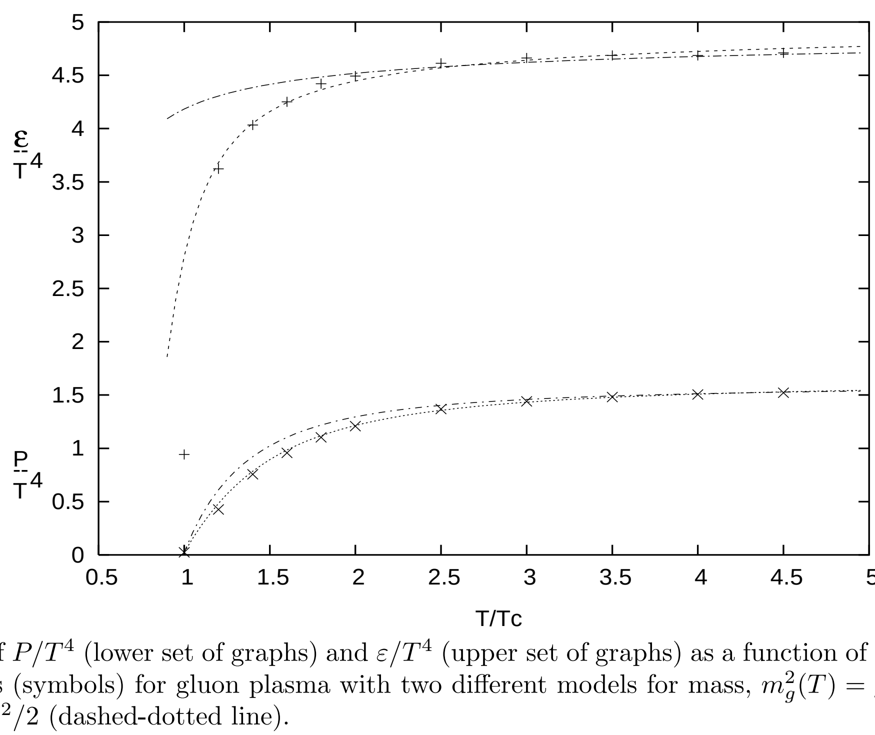


Figure 1: Plots of $P/T^4$ (lower set of graphs) and $\varepsilon/T^4$ (upper set of graphs) as a function of $T/T_c$ from our model and lattice results (symbols) for gluon plasma with two different models for mass, $m_g^2(T) = g^2T^2/3$ (dashed line) and $m_g^2(T) = g^2T^2/2$ (dashed-dotted line).

and results are plotted in Fig. 1, for two different mass terms, $m_g^2(T) = \omega_p^2 = g^2(T)\,T^2\,/3$ (our model) and $m_g^2(T) = g^2(T)\,T^2\,/2$ (other qQGP models). $g^2(T)$ is related to the two-loop order running coupling constant, given by,

$$\alpha_s(T) = \frac{6\pi}{(33-2n_f)\ln(T/\Lambda_T)}\left(1 - \frac{3(153-19n_f)}{(33-2n_f)^2}\frac{\ln(2\ln(T/\Lambda_T))}{\ln(T/\Lambda_T)}\right) , \tag{18}$$

where $\Lambda_T$ is a parameter related to QCD scale parameter. This choice of $\alpha_s(T)$ is motivated from lattice simulations. Then the pressure is obtained from the TD relation Eq. (5) or Eq. (11). Since we have only one parameter to adjust, we don't get good fit for the generally used second choice of quasi-gluon mass. The best fitted parameter is $\Lambda_T/T_c = 0.3$. But a remarkably good fit may be obtained for our choice of gluon mass which is motivated from the fact that the quasi-gluons are the thermal excitations of plasma waves with mass equal to the plasma frequency [10]. The value of the fitted parameter is $\Lambda_T/T_c = 0.65$.

It is interesting to compare our results with the other qQGP models, for example Ref. [6], where $B(T)$ is not zero, but is determined by their TD consistency relation.

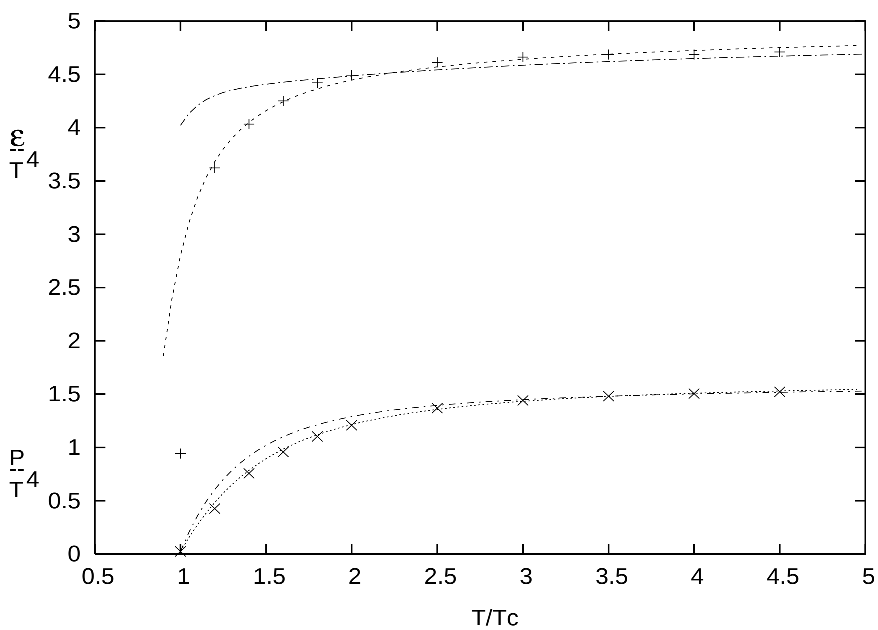


Figure 2: Plots of $P/T^4$ (lower set of graphs) and $\varepsilon/T^4$ (upper set of graphs) as a function of $T/T_c$ from our model and lattice results (symbols) for gluon plasma with $B(T) = 0$ (dashed line) and using so called TD consistency relation (dashed-dotted line).

From the Fig. 2, we see that only at large $T/T_c$ both the results almost match, but differ near to $T/T_c = 1$. We used the same $\alpha_s(T)$ with $\Lambda_T/T_c = .65$ for both the cases. Further, results from our model with $B(T) = 0$ fits well the lattice data. Of course, a very good fit to lattice data was also obtained in Ref. [6], but they used different expression for $\alpha_s(T)$ with two free parameters. They had also another additional parameter related to degrees of freedom. By adjusting these extra two parameters they were able to get good results.

## 5 Conclusions:

Here we pointed out the basic reason for the thermodynamic inconsistency of the extensively studied quasiparticle QGP models [1] and it's consequence of the reformulation of statistical mechanics [5]. Then we have proposed a new qQGP model which very naturally follows from the standard SM and has no TD inconsistency. When we extend our formalism to a system with temperature dependent vacuum energy, again, we get a TD consistent general model and we obtained other widely studied qQGP models as a special case of our model under certain restrictive condition which they called as TD consistency relation. Further, we applied our model, which has only one parameter, to gluon plasma and a remarkable good fit to the LGT data by adjusting just one parameter. Whereas we know that the other qQGP models has 3 or more parameters. Further extension of our model to flavored QGP without and with masses, and also without and with chemical potential, fit remarkably well the lattice results and were reported in [11, 12].

## References


[1] A. Peshier, B. Kampfer, O. P. Pavlenko and G. Soff, Phys. Lett. **B337**, 235 (1994).

[2] D. H. Rischke, M. I. Gorenstein, A. Schafer, H. Stocker and W. Greiner Phys. Lett. **B278**, 19 (1992); Z. Phys. **C56**, 325 (1992).

[3] G. Boyd, J. Engels, F. Karsch, E. Laermann, C. Legeland, M. Lutgemeier and B. Petersson, Phys. Rev. Lett. **75**, 4169 (1995); Nucl. Phys. **B469**, 419 (1996).

[4] V. M. Bannur, Eur. Phys. J. **C11**, 169 (1999).

[5] M. I. Gorenstein and S. N. Yang, Phys. Rev. **D52**, 5206 (1995).

[6] A. Peshier, B. Kampfer, O. P. Pavlenko and G. Soff, Phys. Rev. **D54**, 2399 (1996).

[7] P.Levai and U. Heinz, Phys. Rev. **C57**, 1879 (1998).

[8] R. A. Schneider and W. Weise, Phys. Rev. **C64**, 055201 (2001).

[9] R. K. Pathria, *Statistical Mechanics*, Butterworth-Heinemann, Oxford (1997).

[10] M. V. Medvedev, Phys. Rev. **E59**, R4766 (1999).

[11] V. M. Bannur, hep-ph/0508069.

[12] V. M. Bannur, hep-ph/0604158.